01

Edge ideals of circulant graphs

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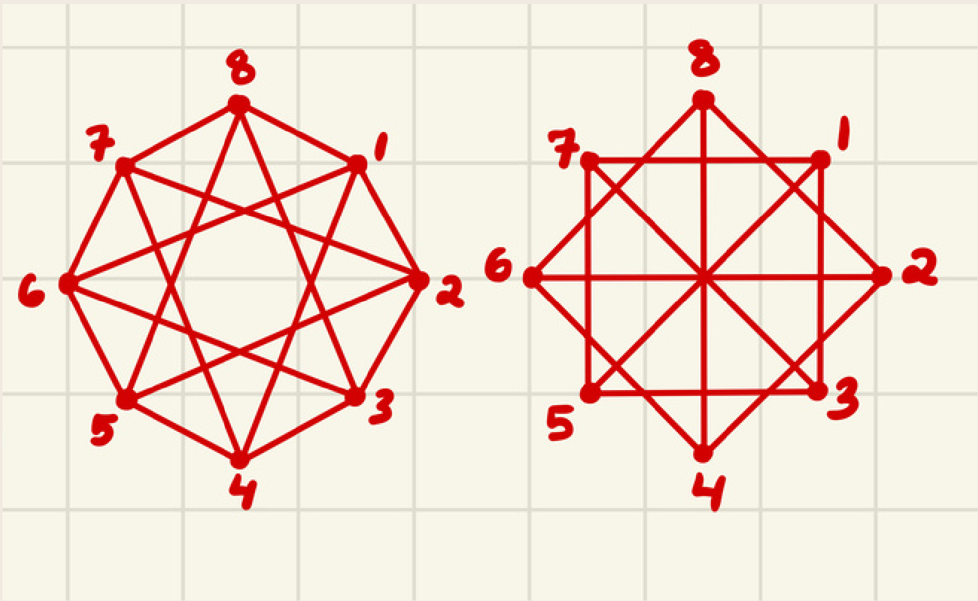
Welcome to class!

Project Proposal

This project aims to look at circulant graphs, which can be described as finite simple graphs on which a cyclic group acts transitively on the vertices. Examples of circulant graphs thus include the n-gons, as well as the complete graphs. To every finite graph G on the vertex set $[n] = \{1, 2, \dots, n\}$, one can associate a monomial ideal I_G in the polynomial ring $S = k[x_1, x_2, \dots, x_n]$. This ideal is generated by the monomials x_i and x_j, where $\{i, j\}$ is an edge in G. Such ideals have attracted a lot of attention in commutative algebra for a long time, where researchers have linked algebraic properties of I_G to graph-theoretic properties of G. In particular we will study the Betti numbers of I_G in the case where G is a circulant graph which is invariant under the dihedral group D_n, and investigate the decomposition of its homologies in terms of the irreducible representations of D_n. This investigation will make use of the computer package Macaulay2 for computing the homologies of the ideals I_G.

Chordal circulant graphs



The first steps

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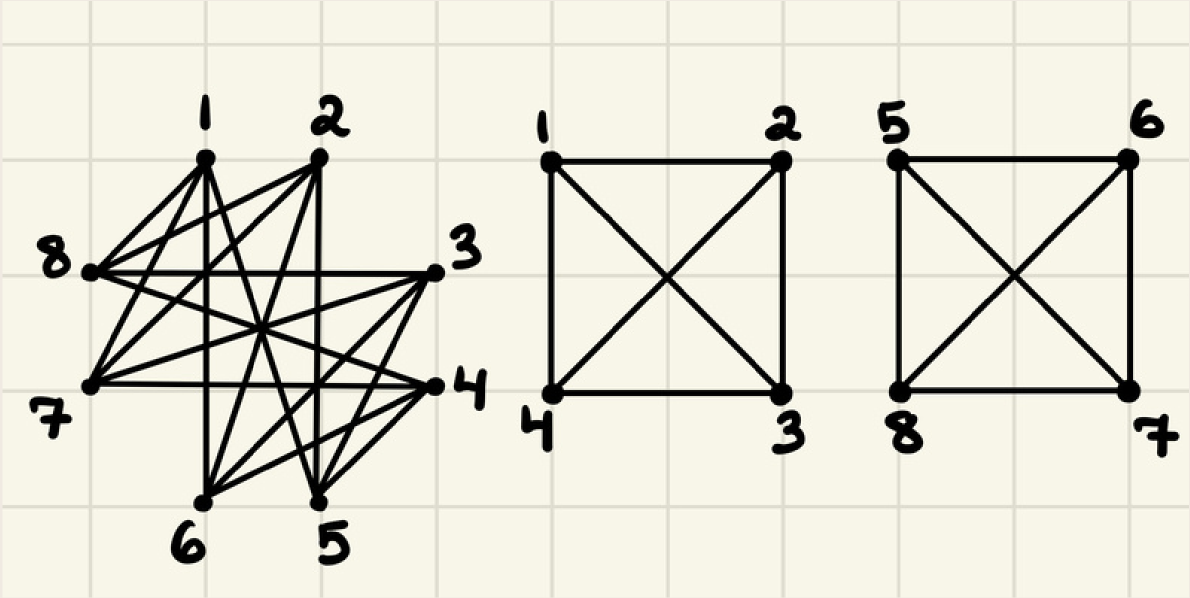
Finite simple graphs on which a cyclic group acts transitively on the vertices.

All cycles of length at least four have a chord, separate from the cycle, connecting two vertices within the cycle.



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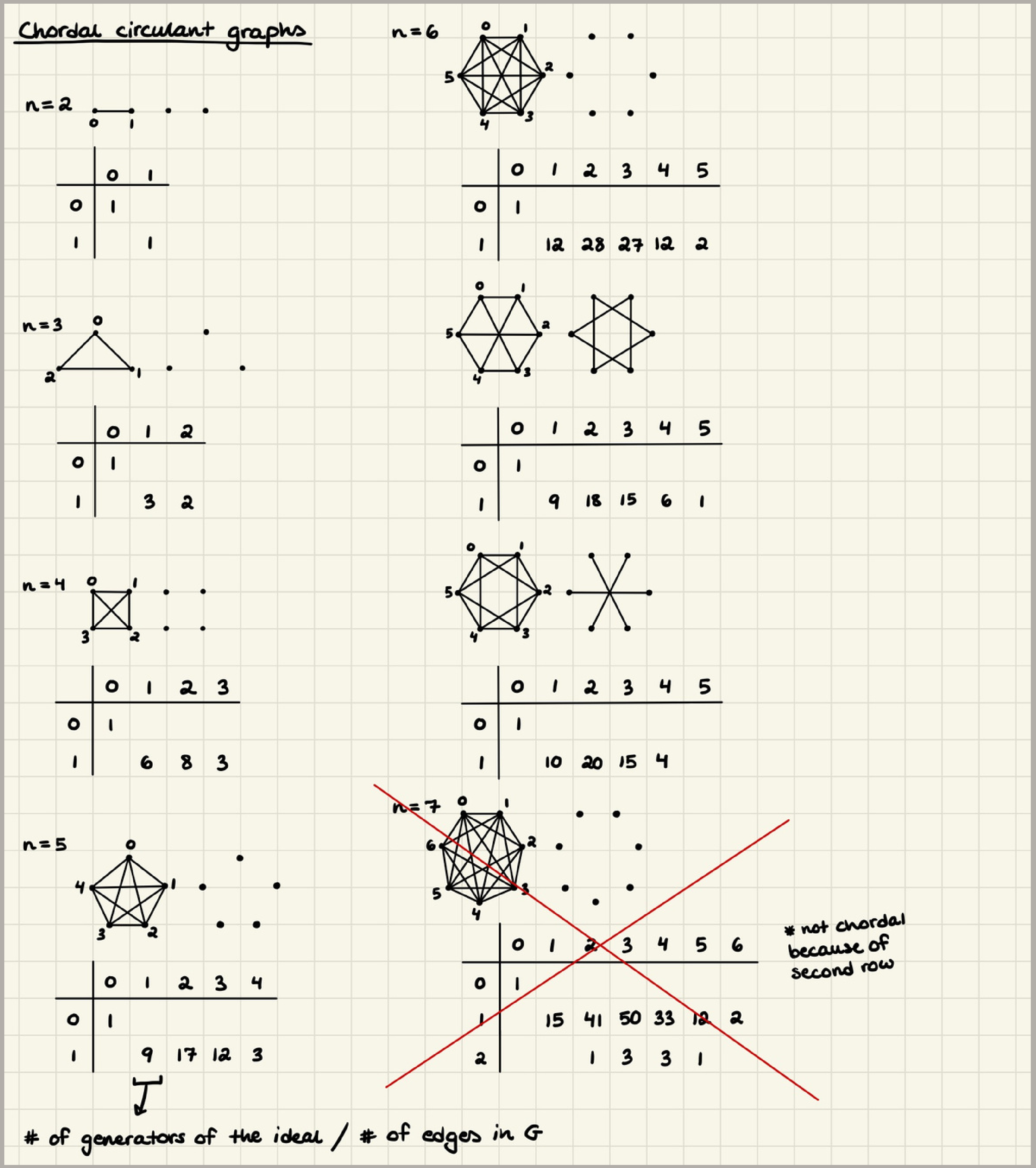
Betti Tables

Sum of the number of connected components of size-i in the compliment graph, minus one!

Counts the basis elements of the linear free resolution in various dimensions - ultimately generated by the monomial ideals of the compliment graph.



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Perfect Elimination Order

Induced cliques

An ordering of vertices such that, for each vertex, its neighbors form a complete induced subgraph.

A graph is chordal if and only if it has a perfect elimination order. Equivalently, if it has a linear free resolution.

Input: 1, 2, 3, 4, 5, 6

$\Sigma = \{\{1,2,3,4,5,6\}\}$ and $i=7$

$\Sigma = \{\{2,3,4,5,6\}\}$ and $i=6$

$v_i = 1 \rightarrow \Sigma = \{\{3,5\},\{2,4,6\}\}$

$\Sigma = \{\{3,5\},\{2,4,6\}\}$ and $i=5$

$v_i = 5 \rightarrow \Sigma = \{\{3\},\{2,4,6\}\}$

$\Sigma = \{\{5\},\{2,4,6\}\}$ and $i=4$

$v_i = 3 \rightarrow \Sigma = \{\{2,4,6\}\}$

$\Sigma = \{\{2,4,6\}\}$ and $i=3$

$v_i = 2 \rightarrow \Sigma = \{\{4,6\}\}$

$\Sigma = \{\{4,6\}\}$ and $i=2$

$v_i = 4 \rightarrow \Sigma = \{\{6\}\}$

$\Sigma = \{\{6\}\}$ and $i=1$

$v_i = 6 \rightarrow \Sigma = \{\{\emptyset\}\}$

Output: 6, 4, 2, 3, 5, 1

Input: 1, 2, 3, 4, 5, 6

$\Sigma = \{\{1,2,3,4,5,6\}\}$ and $i=7$

$\Sigma = \{\{1,2,3,5,6\}\}$ and $i=6$

$v_i = 4 \rightarrow \Sigma = \{\{2,6\},\{1,3,5\}\}$

$\Sigma = \{\{2,6\},\{1,3,5\}\}$ and $i=5$

$v_i = 6 \rightarrow \Sigma = \{\{2\},\{1,3,5\}\}$

$\Sigma = \{\{2\},\{1,3,5\}\}$ and $i=4$

$v_i = 2 \rightarrow \Sigma = \{\{1,3,5\}\}$

$\Sigma = \{\{1,3,5\}\}$ and $i=3$

$v_i = 1 \rightarrow \Sigma = \{\{3,5\}\}$

$\Sigma = \{\{3,5\}\}$ and $i=2$

$v_i = 3 \rightarrow \Sigma = \{\{5\}\}$

$\Sigma = \{\{5\}\}$ and $i=1$

$v_i = 5 \rightarrow \Sigma = \{\{\emptyset\}\}$

Output: 5, 3, 1, 2, 6, 4

More on perfect elimination orders

The algorithm given in Chen (2010) produces the maximum of $k!(m!)^k$ perfect elimination orderings, in which m is the number of vertices in each disjoint connected component of the compliment, and k is the number of disjoint connected components in the compliment.

Algorithm 2.2 from "Minimal free resolutions of linear edge ideals" (Chen, 2010)

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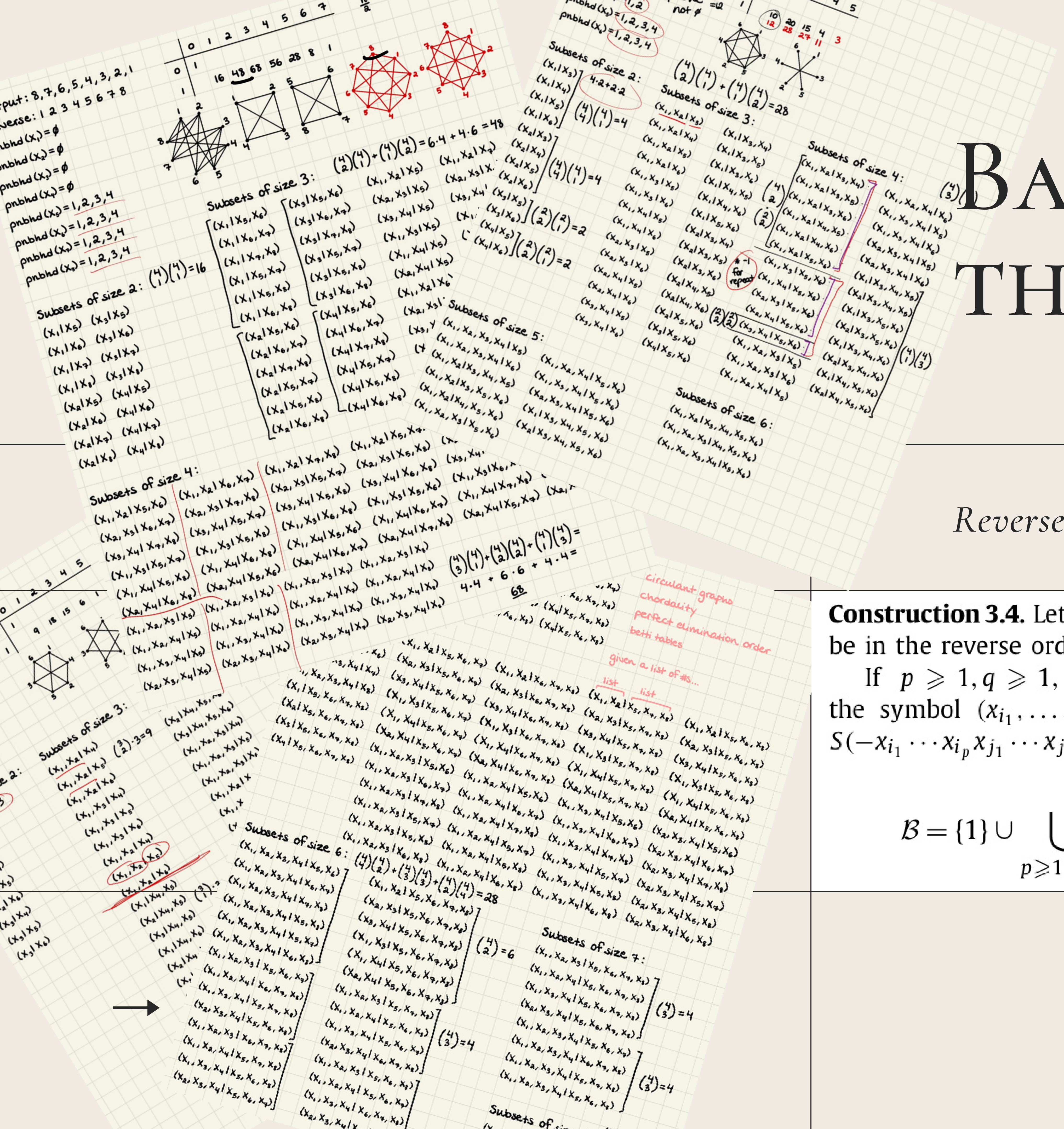
Algorithm 2.2. Let H be a chordal graph with vertices $x_1, \ldots, x_n$. Let Σ be a set containing a sequence of sets.
Input: $\Sigma = \{\{x_1, \ldots, x_n\}\}$, $i = n+1$.
Step 1: Choose and remove a vertex v from the first set in Σ. Set $i := i-1$ and $v_i := v$. If the first set in Σ is now empty, remove it from Σ. Go to setp 2.
Step 2: If $\Sigma = \emptyset$, stop. If $\Sigma \neq \emptyset$, suppose $\Sigma = \{S_1, S_2, \ldots, S_r\}$. For any $1 \leqslant j \leqslant r$, replace the set S_j by two sets T_j and T'_j such that $S_j = T_j \cup T'_j$, $T_j \cap T'_j = \emptyset$, $v_i w \in H$ for any $w \in T_j$ and $v_i w' \notin H$ for any $w' \in T'_j$. Now we set

$$\Sigma := \{T_1, T_2, \ldots, T_r, T'_1, T'_2, \ldots, T'_r\}.$$

Remove all the empty sets from Σ. Go back to step 1.
Output: $v_1, \ldots, v_n$.



Basis Elements of the Free Resolution

Reverse Perfect Elimination Order + Preneighborhoods

Construction 3.4. Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 2.2.

If $p \geqslant 1, q \geqslant 1, 1 \leqslant i_1 < \cdots < i_p < j_1 < \cdots < j_q \leqslant n$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq pnbhd(x_{j_1})$, then the symbol $(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q})$ will be used to denote the generator of the free S-module $S(-x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q})$ in homological degree $p+q-1$ and multidegree $x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q}$. We set

$$\mathcal{B} = \{1\} \cup \bigcup_{p \geqslant 1, q \geqslant 1} \left\{ (x_{i_1}, \ldots, x_{i_p} \mid x_{j_1}, \ldots, x_{j_q}) : \begin{array}{l} 1 \leqslant i_1 < \cdots < i_p < j_1 < \cdots < j_q \leqslant n, \\ \{x_{i_1}, \ldots, x_{i_p}\} \subseteq pnbhd(x_{j_1}) \end{array} \right\}.$$

The process so far

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Fill in the Betti Table	*Construct All Possible "Chen-Symbols" for a given Perfect Elimination Order*	*Relate Betti Table with "Chen-Symbols"*

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i1 : kk = ZZ/101

o1 = kk

o1 : QuotientRing

i2 : S = kk[x_0,x_1,x_2]

o2 = S

o2 : PolynomialRing

i3 : I = ideal (x_0*x_1,x_1*x_2,x_2*x_0)

o3 = ideal (x x , x x , x x )
             0 1   1 2   0 2

o3 : Ideal of S

i4 : M = S/I

o4 = M

o4 : QuotientRing

i5 : M = S^1/I

o5 = cokernel | x_0x_1 x_1x_2 x_0x_2 |
                                 1
o5 : S-module, quotient of S

i6 : Mres = M

o6 = cokernel | x_0x_1 x_1x_2 x_0x_2 |
                                 1
o6 : S-module, quotient of S

i7 : Ires = res I

      1      3      2
o7 = S  <-- S  <-- S  <-- 0

     0      1      2      3

o7 : ChainComplex

i8 : betti Ires

            0 1 2
o8 = total: 1 3 2
         0: 1 . .
         1: . 3 2

o8 : BettiTally

i9 :
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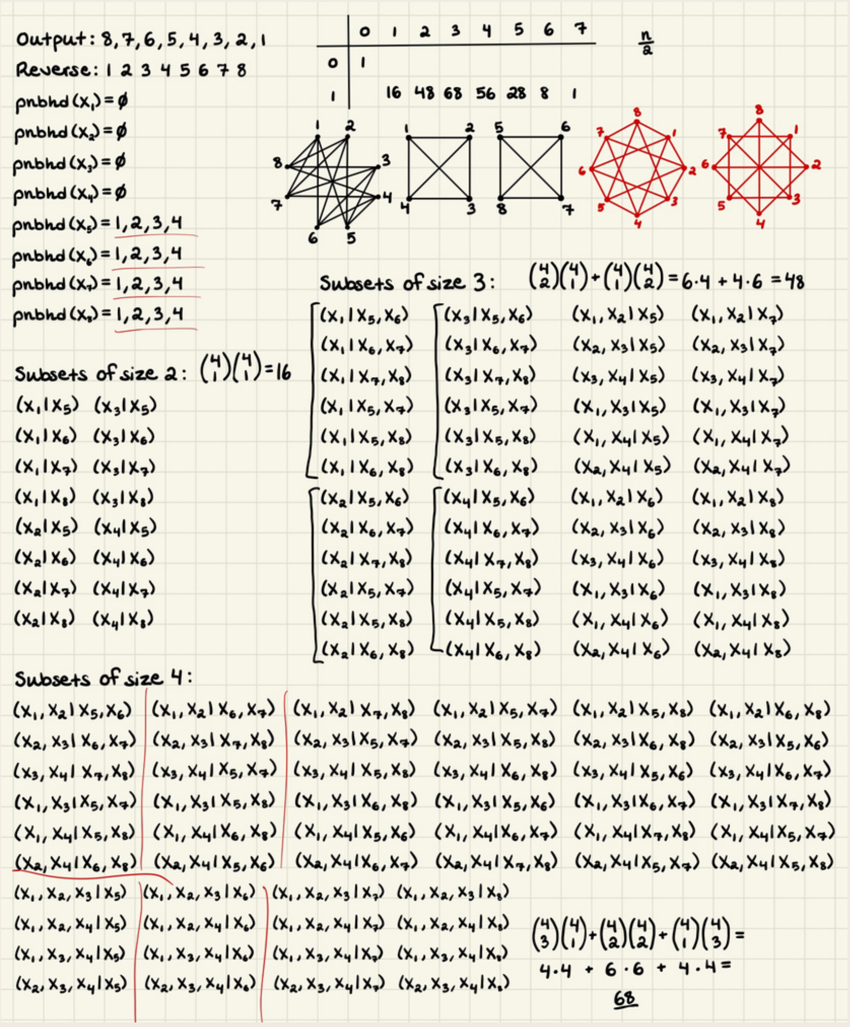

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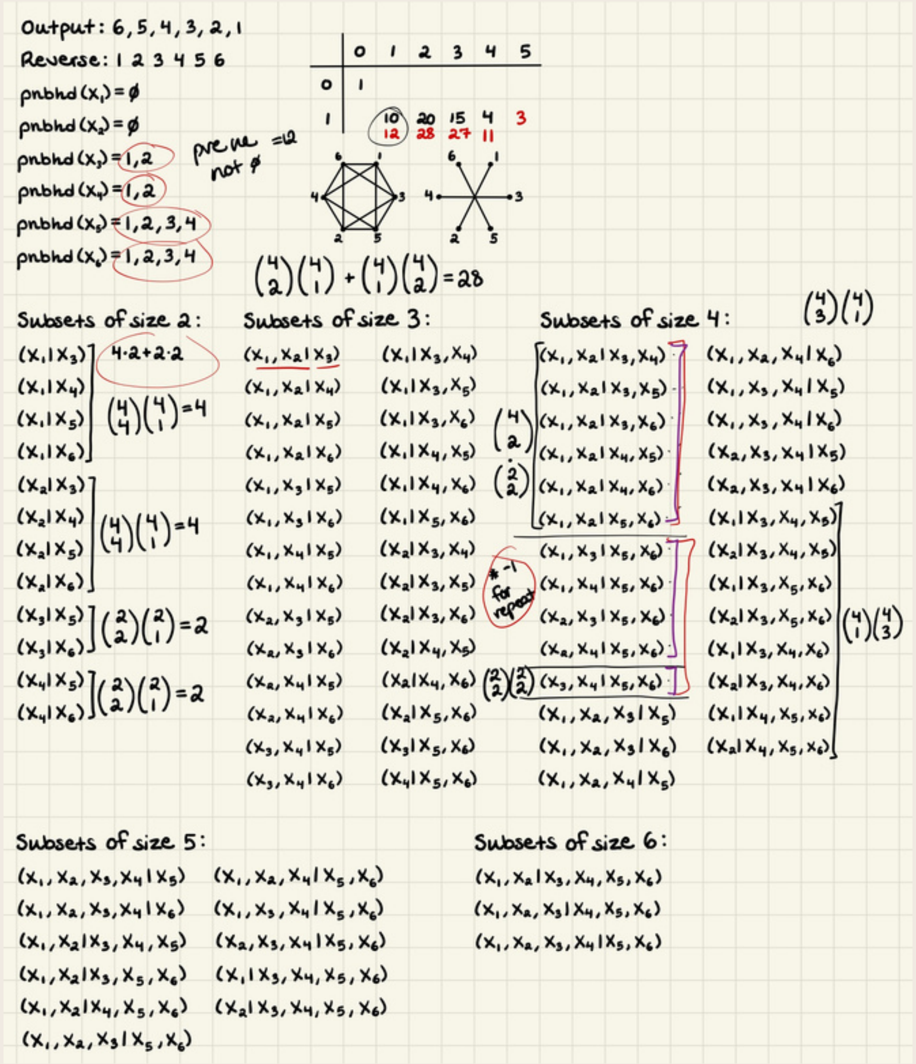
Formula for counting basis elements, aka Betti numbers

For a graph on n vertices such that its compliment is the union of two disjoint complete graphs on $n/2$ vertices, the i^{th} Betti number can be computed as follows:

$$\sum \binom{n/2}{m}\binom{n/2}{k}, \text{ where } m+k=i \text{ and } m,k \geq i-\frac{n}{2}.$$



Next Steps

Write out formal proof for my formula, work out and prove other formulas for graphs with compliments comprised of more than two disjoint components, write a code for computing basis elements.

Is everything clear?