

EDGE IDEALS OF CIRCULANT GRAPHS

Dr. Emil Skoldberg, Sasha Northrup



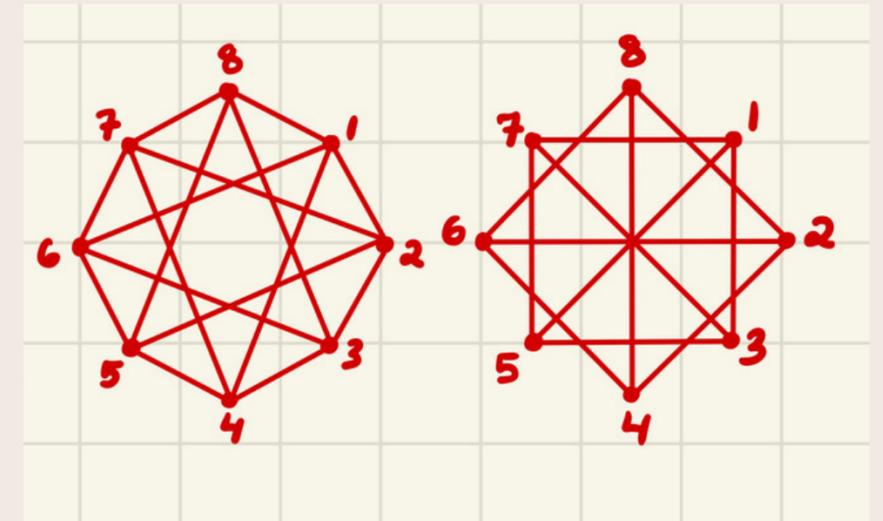
WELCOME TO CLASS!

Project Proposal

This project aims to look at circulant graphs, which can be described as finite simple graphs on which a cyclic group acts transitively on the vertices. Examples of circulant graphs thus include the n -gons, as well as the complete graphs. To every finite graph G on the vertex set $[n] = \{1, 2, \dots, n\}$, one can associate a monomial ideal \mathbf{I}_G in the polynomial ring $S = k[x_1, x_2, \dots, x_n]$. This ideal is generated by the monomials x_i and x_j , where $\{i, j\}$ is an edge in G . Such ideals have attracted a lot of attention in commutative algebra for a long time, where researchers have linked algebraic properties of \mathbf{I}_G to graph-theoretic properties of G . In particular we will study the Betti numbers of \mathbf{I}_G in the case where G is a circulant graph which is invariant under the dihedral group D_n , and investigate the decomposition of its homologies in terms of the irreducible representations of D_n . This investigation will make use of the computer package Macaulay2 for computing the homologies of the ideals \mathbf{I}_G .



CHORDAL CIRCULANT GRAPHS



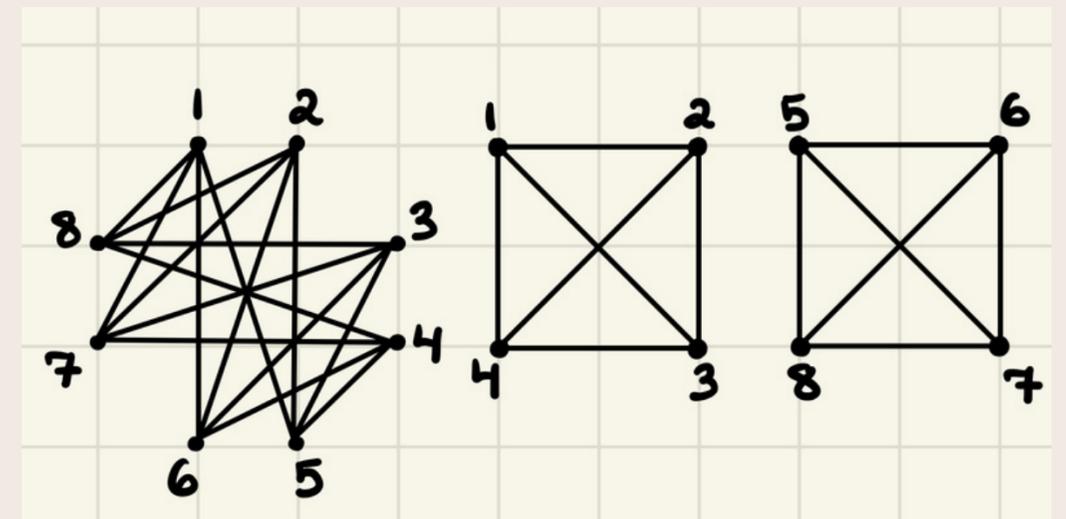
The first steps



03.

Finite simple graphs on which a cyclic group acts transitively on the vertices.

All cycles of length at least four have a chord, separate from the cycle, connecting two vertices within the cycle.



BETTI TABLES

Sum of the number of connected components of size- i in the complement graph, minus one!

Counts the basis elements of the linear free resolution in various dimensions - ultimately generated by the monomial ideals of the complement graph.



Chordal circulant graphs

$n=2$

| | | |
|---|---|---|
| | 0 | 1 |
| 0 | 1 | |
| 1 | | 1 |

$n=3$

| | | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| 0 | 1 | | |
| 1 | | 3 | 2 |

$n=4$

| | | | | |
|---|---|---|---|---|
| | 0 | 1 | 2 | 3 |
| 0 | 1 | | | |
| 1 | | 6 | 8 | 3 |

$n=5$

| | | | | | |
|---|---|---|----|----|---|
| | 0 | 1 | 2 | 3 | 4 |
| 0 | 1 | | | | |
| 1 | | 9 | 17 | 12 | 3 |

$n=6$

| | | | | | | |
|---|---|----|----|----|----|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | | | | | |
| 1 | | 12 | 28 | 27 | 12 | 2 |

$n=6$

| | | | | | | |
|---|---|---|----|----|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | | | | | |
| 1 | | 9 | 18 | 15 | 6 | 1 |

$n=4$

| | | | | | | |
|---|---|----|----|----|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | | | | | |
| 1 | | 10 | 20 | 15 | 4 | |

~~$n=7$~~

| | | | | | | | |
|---|---|----|----|----|----|----|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | | | | | | |
| 1 | | 15 | 41 | 50 | 33 | 12 | 2 |
| 2 | | | 1 | 3 | 3 | 1 | |

not chordal because of second row

of generators of the ideal / # of edges in G

Input: 1, 2, 3, 4, 5, 6

$\Sigma = \{\{1, 2, 3, 4, 5, 6\}\}$ and $i=7$

$\Sigma = \{\{2, 3, 4, 5, 6\}\}$ and $i=6$

$v_i=1 \rightarrow \Sigma = \{\{3, 5\}, \{2, 4, 6\}\}$

$\Sigma = \{\{3, 5\}, \{2, 4, 6\}\}$ and $i=5$

$v_i=5 \rightarrow \Sigma = \{\{3\}, \{2, 4, 6\}\}$

$\Sigma = \{\{5\}, \{2, 4, 6\}\}$ and $i=4$

$v_i=3 \rightarrow \Sigma = \{\{2, 4, 6\}\}$

$\Sigma = \{\{2, 4, 6\}\}$ and $i=3$

$v_i=2 \rightarrow \Sigma = \{\{4, 6\}\}$

$\Sigma = \{\{4, 6\}\}$ and $i=2$

$v_i=4 \rightarrow \Sigma = \{\{6\}\}$

$\Sigma = \{\{6\}\}$ and $i=1$

$v_i=6 \rightarrow \Sigma = \{\{\emptyset\}\}$

Output: 6, 4, 2, 3, 5, 1

Input: 1, 2, 3, 4, 5, 6

$\Sigma = \{\{1, 2, 3, 4, 5, 6\}\}$ and $i=7$

$\Sigma = \{\{1, 2, 3, 5, 6\}\}$ and $i=6$

$v_i=4 \rightarrow \Sigma = \{\{2, 6\}, \{1, 3, 5\}\}$

$\Sigma = \{\{2, 6\}, \{1, 3, 5\}\}$ and $i=5$

$v_i=6 \rightarrow \Sigma = \{\{2\}, \{1, 3, 5\}\}$

$\Sigma = \{\{2\}, \{1, 3, 5\}\}$ and $i=4$

$v_i=2 \rightarrow \Sigma = \{\{1, 3, 5\}\}$

$\Sigma = \{\{1, 3, 5\}\}$ and $i=3$

$v_i=1 \rightarrow \Sigma = \{\{3, 5\}\}$

$\Sigma = \{\{3, 5\}\}$ and $i=2$

$v_i=3 \rightarrow \Sigma = \{\{5\}\}$

$\Sigma = \{\{5\}\}$ and $i=1$

$v_i=5 \rightarrow \Sigma = \{\{\emptyset\}\}$

Output: 5, 3, 1, 2, 6, 4

PERFECT ELIMINATION ORDER

Induced cliques

An ordering of vertices such that, for each vertex, its neighbors form a complete induced subgraph.

A graph is chordal if and only if it has a perfect elimination order. Equivalently, if it has a linear free resolution.



MORE ON PERFECT ELIMINATION ORDERS

The algorithm given in Chen (2010) produces the **maximum** of $k!(m!)^k$ perfect elimination orderings, in which m is the number of vertices in each disjoint connected component of the compliment, and k is the number of disjoint connected components in the compliment.

Algorithm 2.2 from "Minimal free resolutions of linear edge ideals"
(Chen, 2010)



Algorithm 2.2. Let H be a chordal graph with vertices x_1, \dots, x_n . Let Σ be a set containing a sequence of sets.

Input: $\Sigma = \{\{x_1, \dots, x_n\}\}$, $i = n + 1$.

Step 1: Choose and remove a vertex v from the first set in Σ . Set $i := i - 1$ and $v_i := v$. If the first set in Σ is now empty, remove it from Σ . Go to step 2.

Step 2: If $\Sigma = \emptyset$, stop. If $\Sigma \neq \emptyset$, suppose $\Sigma = \{S_1, S_2, \dots, S_r\}$. For any $1 \leq j \leq r$, replace the set S_j by two sets T_j and T'_j such that $S_j = T_j \cup T'_j$, $T_j \cap T'_j = \emptyset$, $v_i w \in H$ for any $w \in T_j$ and $v_i w' \notin H$ for any $w' \in T'_j$. Now we set

$$\Sigma := \{T_1, T_2, \dots, T_r, T'_1, T'_2, \dots, T'_r\}.$$

Remove all the empty sets from Σ . Go back to step 1.

Output: v_1, \dots, v_n .

THE PROCESS SO FAR

O1 Fill in the Betti Table

```

i1 : kk = ZZ/101
o1 = kk
o1 : QuotientRing
i2 : S = kk[x_0,x_1,x_2]
o2 = S
o2 : PolynomialRing
i3 : I = ideal (x_0*x_1,x_1*x_2,x_2*x_0)
o3 = ideal (x_0 x_1 x_2
           0 1 1 2 0 2)
o3 : Ideal of S
i4 : M = S/I
o4 = M
o4 : QuotientRing
i5 : M = S^1/I
o5 = cokernel | x_0x_1 x_1x_2 x_0x_2 |
           1
o5 : S-module, quotient of S
i6 : Mres = M
o6 = cokernel | x_0x_1 x_1x_2 x_0x_2 |
           1
o6 : S-module, quotient of S
i7 : Ires = res I
o7 = S 1 3 2
    0 1 2 3
o7 : ChainComplex
i8 : betti Ires
o8 = total: 1 3 2
      0: 1 . .
      1: . 3 2
o8 : BettiTable
i9 :

```

O2 Construct All Possible "Chen-Symbols" for a given Perfect Elimination Order

O3 Relate Betti Table with "Chen-Symbols"

Output: 8,7,6,5,4,3,2,1
Reverse: 1 2 3 4 5 6 7 8

pnbhd(x₀) = ∅
pnbhd(x₁) = ∅
pnbhd(x₂) = ∅
pnbhd(x₃) = ∅
pnbhd(x₄) = 1,2,3,4
pnbhd(x₅) = 1,2,3,4
pnbhd(x₆) = 1,2,3,4
pnbhd(x₇) = 1,2,3,4

Subsets of size 2: $\binom{4}{1}\binom{4}{1} = 16$

| | |
|------------------------------------|------------------------------------|
| (x ₁ , x ₅) | (x ₃ , x ₅) |
| (x ₁ , x ₆) | (x ₃ , x ₆) |
| (x ₁ , x ₇) | (x ₃ , x ₇) |
| (x ₁ , x ₈) | (x ₃ , x ₈) |
| (x ₂ , x ₅) | (x ₄ , x ₅) |
| (x ₂ , x ₆) | (x ₄ , x ₆) |
| (x ₂ , x ₇) | (x ₄ , x ₇) |
| (x ₂ , x ₈) | (x ₄ , x ₈) |
| (x ₅ , x ₆) | (x ₅ , x ₇) |
| (x ₅ , x ₈) | (x ₆ , x ₇) |
| (x ₅ , x ₈) | (x ₆ , x ₈) |
| (x ₆ , x ₇) | (x ₇ , x ₈) |

Subsets of size 3: $\binom{4}{2}\binom{4}{1} + \binom{4}{1}\binom{4}{2} = 6 \cdot 4 + 4 \cdot 6 = 48$

| | | | |
|---|---|---|---|
| (x ₁ , x ₅ , x ₆) | (x ₃ , x ₅ , x ₆) | (x ₁ , x ₂ , x ₅) | (x ₁ , x ₂ , x ₆) |
| (x ₁ , x ₅ , x ₇) | (x ₃ , x ₅ , x ₇) | (x ₂ , x ₃ , x ₅) | (x ₂ , x ₃ , x ₆) |
| (x ₁ , x ₅ , x ₈) | (x ₃ , x ₅ , x ₈) | (x ₃ , x ₄ , x ₅) | (x ₃ , x ₄ , x ₆) |
| (x ₁ , x ₆ , x ₇) | (x ₃ , x ₆ , x ₇) | (x ₁ , x ₃ , x ₅) | (x ₁ , x ₃ , x ₆) |
| (x ₁ , x ₆ , x ₈) | (x ₃ , x ₆ , x ₈) | (x ₂ , x ₄ , x ₅) | (x ₂ , x ₄ , x ₆) |
| (x ₁ , x ₇ , x ₈) | (x ₃ , x ₇ , x ₈) | (x ₁ , x ₂ , x ₆) | (x ₁ , x ₂ , x ₇) |
| (x ₂ , x ₅ , x ₆) | (x ₄ , x ₅ , x ₆) | (x ₂ , x ₃ , x ₅) | (x ₂ , x ₃ , x ₆) |
| (x ₂ , x ₅ , x ₇) | (x ₄ , x ₅ , x ₇) | (x ₃ , x ₄ , x ₅) | (x ₃ , x ₄ , x ₆) |
| (x ₂ , x ₅ , x ₈) | (x ₄ , x ₅ , x ₈) | (x ₁ , x ₃ , x ₆) | (x ₁ , x ₃ , x ₇) |
| (x ₂ , x ₆ , x ₇) | (x ₄ , x ₆ , x ₇) | (x ₂ , x ₄ , x ₅) | (x ₂ , x ₄ , x ₆) |
| (x ₂ , x ₆ , x ₈) | (x ₄ , x ₆ , x ₈) | (x ₁ , x ₂ , x ₇) | (x ₁ , x ₂ , x ₈) |
| (x ₂ , x ₇ , x ₈) | (x ₄ , x ₇ , x ₈) | (x ₃ , x ₄ , x ₆) | (x ₃ , x ₄ , x ₇) |
| (x ₃ , x ₅ , x ₆) | (x ₄ , x ₅ , x ₆) | (x ₁ , x ₃ , x ₇) | (x ₁ , x ₃ , x ₈) |
| (x ₃ , x ₅ , x ₇) | (x ₄ , x ₅ , x ₇) | (x ₂ , x ₄ , x ₇) | (x ₂ , x ₄ , x ₈) |
| (x ₃ , x ₅ , x ₈) | (x ₄ , x ₅ , x ₈) | (x ₁ , x ₄ , x ₅) | (x ₁ , x ₄ , x ₆) |
| (x ₃ , x ₆ , x ₇) | (x ₄ , x ₆ , x ₇) | (x ₂ , x ₄ , x ₆) | (x ₂ , x ₄ , x ₇) |
| (x ₃ , x ₆ , x ₈) | (x ₄ , x ₆ , x ₈) | (x ₁ , x ₄ , x ₇) | (x ₁ , x ₄ , x ₈) |
| (x ₃ , x ₇ , x ₈) | (x ₄ , x ₇ , x ₈) | (x ₂ , x ₄ , x ₈) | (x ₂ , x ₄ , x ₉) |

Subsets of size 4: $\binom{4}{3}\binom{4}{1} + \binom{4}{2}\binom{4}{2} + \binom{4}{1}\binom{4}{3} = 4 \cdot 4 + 6 \cdot 6 + 4 \cdot 4 = 68$

| | | | | | |
|--|--|--|--|--|--|
| (x ₁ , x ₂ , x ₅ , x ₆) | (x ₁ , x ₂ , x ₅ , x ₇) | (x ₁ , x ₂ , x ₅ , x ₈) | (x ₁ , x ₂ , x ₆ , x ₇) | (x ₁ , x ₂ , x ₆ , x ₈) | (x ₁ , x ₂ , x ₇ , x ₈) |
| (x ₁ , x ₂ , x ₆ , x ₇) | (x ₁ , x ₂ , x ₆ , x ₈) | (x ₁ , x ₂ , x ₇ , x ₈) | (x ₁ , x ₃ , x ₅ , x ₆) | (x ₁ , x ₃ , x ₅ , x ₇) | (x ₁ , x ₃ , x ₅ , x ₈) |
| (x ₁ , x ₃ , x ₅ , x ₆) | (x ₁ , x ₃ , x ₅ , x ₇) | (x ₁ , x ₃ , x ₅ , x ₈) | (x ₁ , x ₃ , x ₆ , x ₇) | (x ₁ , x ₃ , x ₆ , x ₈) | (x ₁ , x ₃ , x ₇ , x ₈) |
| (x ₁ , x ₃ , x ₆ , x ₇) | (x ₁ , x ₃ , x ₆ , x ₈) | (x ₁ , x ₃ , x ₇ , x ₈) | (x ₁ , x ₄ , x ₅ , x ₆) | (x ₁ , x ₄ , x ₅ , x ₇) | (x ₁ , x ₄ , x ₅ , x ₈) |
| (x ₁ , x ₄ , x ₅ , x ₆) | (x ₁ , x ₄ , x ₅ , x ₇) | (x ₁ , x ₄ , x ₅ , x ₈) | (x ₁ , x ₄ , x ₆ , x ₇) | (x ₁ , x ₄ , x ₆ , x ₈) | (x ₁ , x ₄ , x ₇ , x ₈) |
| (x ₁ , x ₄ , x ₆ , x ₇) | (x ₁ , x ₄ , x ₆ , x ₈) | (x ₁ , x ₄ , x ₇ , x ₈) | (x ₂ , x ₃ , x ₅ , x ₆) | (x ₂ , x ₃ , x ₅ , x ₇) | (x ₂ , x ₃ , x ₅ , x ₈) |
| (x ₂ , x ₃ , x ₅ , x ₆) | (x ₂ , x ₃ , x ₅ , x ₇) | (x ₂ , x ₃ , x ₅ , x ₈) | (x ₂ , x ₃ , x ₆ , x ₇) | (x ₂ , x ₃ , x ₆ , x ₈) | (x ₂ , x ₃ , x ₇ , x ₈) |
| (x ₂ , x ₃ , x ₆ , x ₇) | (x ₂ , x ₃ , x ₆ , x ₈) | (x ₂ , x ₃ , x ₇ , x ₈) | (x ₂ , x ₄ , x ₅ , x ₆) | (x ₂ , x ₄ , x ₅ , x ₇) | (x ₂ , x ₄ , x ₅ , x ₈) |
| (x ₂ , x ₄ , x ₅ , x ₆) | (x ₂ , x ₄ , x ₅ , x ₇) | (x ₂ , x ₄ , x ₅ , x ₈) | (x ₂ , x ₄ , x ₆ , x ₇) | (x ₂ , x ₄ , x ₆ , x ₈) | (x ₂ , x ₄ , x ₇ , x ₈) |
| (x ₂ , x ₄ , x ₆ , x ₇) | (x ₂ , x ₄ , x ₆ , x ₈) | (x ₂ , x ₄ , x ₇ , x ₈) | (x ₃ , x ₄ , x ₅ , x ₆) | (x ₃ , x ₄ , x ₅ , x ₇) | (x ₃ , x ₄ , x ₅ , x ₈) |
| (x ₃ , x ₄ , x ₅ , x ₆) | (x ₃ , x ₄ , x ₅ , x ₇) | (x ₃ , x ₄ , x ₅ , x ₈) | (x ₃ , x ₄ , x ₆ , x ₇) | (x ₃ , x ₄ , x ₆ , x ₈) | (x ₃ , x ₄ , x ₇ , x ₈) |
| (x ₃ , x ₄ , x ₆ , x ₇) | (x ₃ , x ₄ , x ₆ , x ₈) | (x ₃ , x ₄ , x ₇ , x ₈) | (x ₃ , x ₅ , x ₆ , x ₇) | (x ₃ , x ₅ , x ₆ , x ₈) | (x ₃ , x ₅ , x ₇ , x ₈) |
| (x ₃ , x ₅ , x ₆ , x ₇) | (x ₃ , x ₅ , x ₆ , x ₈) | (x ₃ , x ₅ , x ₇ , x ₈) | (x ₃ , x ₆ , x ₇ , x ₈) | (x ₃ , x ₆ , x ₈) | (x ₃ , x ₇ , x ₈) |
| (x ₃ , x ₆ , x ₇ , x ₈) | (x ₃ , x ₇ , x ₈) | (x ₄ , x ₅ , x ₆ , x ₇) | (x ₄ , x ₅ , x ₆ , x ₈) | (x ₄ , x ₅ , x ₇ , x ₈) | (x ₄ , x ₅ , x ₈) |
| (x ₄ , x ₅ , x ₆ , x ₇) | (x ₄ , x ₅ , x ₆ , x ₈) | (x ₄ , x ₅ , x ₇ , x ₈) | (x ₄ , x ₆ , x ₇ , x ₈) | (x ₄ , x ₆ , x ₈) | (x ₄ , x ₇ , x ₈) |
| (x ₄ , x ₆ , x ₇ , x ₈) | (x ₄ , x ₇ , x ₈) | (x ₅ , x ₆ , x ₇ , x ₈) | (x ₅ , x ₇ , x ₈) | (x ₅ , x ₈) | (x ₆ , x ₇ , x ₈) |
| (x ₅ , x ₆ , x ₇ , x ₈) | (x ₅ , x ₇ , x ₈) | (x ₆ , x ₇ , x ₈) | (x ₆ , x ₈) | (x ₇ , x ₈) | (x ₈) |



FORMULA FOR COUNTING BASIS ELEMENTS, AKA BETTI NUMBERS



Research Project | May-July 2023

For a graph on n vertices such that its complement is the union of two disjoint complete graphs on $n/2$ vertices, the i^{th} Betti number can be computed as follows:

$$\sum \binom{n/2}{m} \binom{n/2}{k}, \text{ where } m+k=i \text{ and } m, k \geq i - \frac{n}{2}.$$

Next Steps

Write out formal proof for my formula, work out and prove other formulas for graphs with complements comprised of more than two disjoint components, write a code for computing basis elements.

Output: 6, 5, 4, 3, 2, 1
Reverse: 1 2 3 4 5 6

| | | | | | | |
|---|---|----|----|----|----|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | | | | | |
| 1 | | 10 | 20 | 15 | 4 | 3 |
| 2 | | 12 | 28 | 27 | 11 | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |

pnhd(x₁) = ∅
pnhd(x₂) = ∅
pnhd(x₃) = {1, 2} pre-ve = 2
pnhd(x₄) = {1, 2} not ∅
pnhd(x₅) = {1, 2, 3, 4}
pnhd(x₆) = {1, 2, 3, 4}

$\binom{4}{2} \binom{4}{1} + \binom{4}{1} \binom{4}{2} = 28$

Subsets of size 2: $\binom{4}{2} + \binom{4}{2} = 4$
 (x₁, x₂)
 (x₁, x₃)
 (x₁, x₄)
 (x₁, x₅)
 (x₁, x₆)
 (x₂, x₃)
 (x₂, x₄)
 (x₂, x₅)
 (x₂, x₆)
 (x₃, x₄)
 (x₃, x₅)
 (x₃, x₆)
 (x₄, x₅)
 (x₄, x₆)
 (x₅, x₆)

Subsets of size 3: $\binom{4}{2} \binom{4}{1} = 4$
 (x₁, x₂, x₃)
 (x₁, x₂, x₄)
 (x₁, x₂, x₅)
 (x₁, x₂, x₆)
 (x₁, x₃, x₄)
 (x₁, x₃, x₅)
 (x₁, x₃, x₆)
 (x₁, x₄, x₅)
 (x₁, x₄, x₆)
 (x₁, x₅, x₆)
 (x₂, x₃, x₄)
 (x₂, x₃, x₅)
 (x₂, x₃, x₆)
 (x₂, x₄, x₅)
 (x₂, x₄, x₆)
 (x₂, x₅, x₆)
 (x₃, x₄, x₅)
 (x₃, x₄, x₆)
 (x₃, x₅, x₆)
 (x₄, x₅, x₆)

Subsets of size 4: $\binom{4}{3} \binom{4}{1} = 4$
 (x₁, x₂, x₃, x₄)
 (x₁, x₂, x₃, x₅)
 (x₁, x₂, x₃, x₆)
 (x₁, x₂, x₄, x₅)
 (x₁, x₂, x₄, x₆)
 (x₁, x₂, x₅, x₆)
 (x₁, x₃, x₄, x₅)
 (x₁, x₃, x₄, x₆)
 (x₁, x₃, x₅, x₆)
 (x₁, x₄, x₅, x₆)
 (x₂, x₃, x₄, x₅)
 (x₂, x₃, x₄, x₆)
 (x₂, x₃, x₅, x₆)
 (x₂, x₄, x₅, x₆)
 (x₃, x₄, x₅, x₆)

Subsets of size 5: $\binom{4}{2} \binom{4}{2} = 2$
 (x₁, x₂, x₃, x₄, x₅)
 (x₁, x₂, x₃, x₄, x₆)
 (x₁, x₂, x₃, x₅, x₆)
 (x₁, x₂, x₄, x₅, x₆)
 (x₂, x₃, x₄, x₅, x₆)

Subsets of size 6: $\binom{4}{3} \binom{4}{3} = 4$
 (x₁, x₂, x₃, x₄, x₅, x₆)

IS EVERYTHING CLEAR?

